Singularities at a dense set of temperature singularities in the Husimi tree

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Complex temperature singularities of the three-site interacting Ising model on the Husimi tree have been investigated in the presence of magnetic field. At definite values of magnetic field these singularities were shown to lie at a dense set touching the real axis and as a consequence condensation of critical points took place. [S1063-651X(97)11911-0]

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The Ising model with multisite interactions plays an important role in investigations of real physical systems. A recent study of the model on a Husimi tree yielded a qualitatively better approximation for ferromagnetic phase diagrams than those resulting from conventional mean-field theories for multisite interactions [1]. The change of sign of the three-site coupling constant on the Husimi tree creates a drastically different situation, much more so than in the two-site interaction ferromagnetic case. The results for the magnetization obtained at certain values of the interaction constant involve period doubling, chaos, etc., and as a consequence a large variety of phase transitions takes place.

In this paper we investigate Fisher's zeroes of partition function of three-site interacting Ising model on the Husimi tree [1,2] in the presence of magnetic field and show that at certain values of the magnetic field these singularities lie at a dense set. The Husimi tree is characterized by γ, the number of triangles that go out from each site and by n, the number of generations. The three-site interacting Ising model in a magnetic field is defined by the Hamiltonian

$$H=-J_3'\sum_{\triangle}\sigma_i\sigma_j\sigma_k-h'\sum_i\sigma_i, \qquad (1)$$

where σ_i takes values ± 1, the first sum goes over all triangular faces of the Husimi tree and the second over all sites, where $J_3=\beta J_3'$, $h=\beta h'$, $\beta=1/kT$, and J' is the three-site coupling strength, h' is the external magnetic field, T is the temperature of the system. Note that the solution of the three-site interacting Ising model on the triangular lattice in the absence of magnetic field may be obtained by mapping the model to the solvable case of the eight-vertex model on the Kagomé lattice [4].

When the Husimi tree is cut apart at the base site, it separates into γ identical branches. The partition function can be written as follows:

$$Z_n=\sum_{\{\sigma_0\}}\exp\{h\sigma_0\}[g_n(\sigma_0)]^\gamma, \qquad (2)$$

where σ_0 are spins of base site, n is the number of generations ($n\to\infty$ corresponds to the thermodynamic limit). Each branch, in turn, can be cut along any site of the first generation, which is nearest to the central site. The expression for $g_n(\sigma_0)$ can therefore be rewritten in the form

$$g_n(\sigma_0)=\sum_{\{\sigma_1\}}\exp\left\{J_3\sigma_0\sigma_1^{(1)}\sigma_1^{(2)}+h\sum_{j=1,2}\sigma_1^{(j)}\right\}\times[g_{n-1}(\sigma_1^{(1)})]^{\gamma-1}[g_{n-1}(\sigma_1^{(2)})]^{\gamma-1}. \qquad (3)$$

We introduce the following variable:

$$x_n=\frac{g_n(+)}{g_n(-)}. \qquad (4)$$

For x_n we can then obtain the recursion relation

$$x_n=f(x_{n-1}),\quad f(x)=\frac{z\mu^2x^{2(\gamma-1)}+2\mu x^{\gamma-1}+z}{\mu^2x^{2(\gamma-1)}+2z\mu x^{\gamma-1}+1}, \qquad (5)$$

where $z=e^{2J_3}$, $\mu=e^{2h}$, and $0\leqslant x_n\leqslant 1$. The function $f(x)$ is unimodal: it is continuously differentiable, and has one maximum x^* in $[0,1]$. Note that $f(x^*)=1$ for any γ, h, and T. This function is nonhyperbolic (hyperbolicity for one-dimensional maps means that $1<|f'|<\infty$ in all points) and maps the interval $[0,1]$ onto $[z,1]$.

Through x_n, obtained by Eq. (5), one can express the magnetization of the central base site:

$$m_n=\langle\sigma_0\rangle=\frac{\mu x_n^\gamma-1}{\mu x_n^\gamma+1}, \qquad (6)$$

The diagram of the three-site interacting Ising model remains unchanged when $J\to -J$, $h\to -h$, because Hamiltonian (1) involves even terms in σ. Therefore we will consider $h\geqslant 0$. It is noteworthy that in the multisite interaction model the Lee-Yang theorem [11] is irrelevant and the phase transition may occur at $h\neq 0$. At $J>0$ the investigation of the three-site interacting Ising model on the Husimi tree shows good agreement [1] with the phase transition line obtained from self-duality,whereas the conventional mean field approximation fails at low temperatures, but at $J<0$ in the sufficiently low kT limit this model shows very unusual be-

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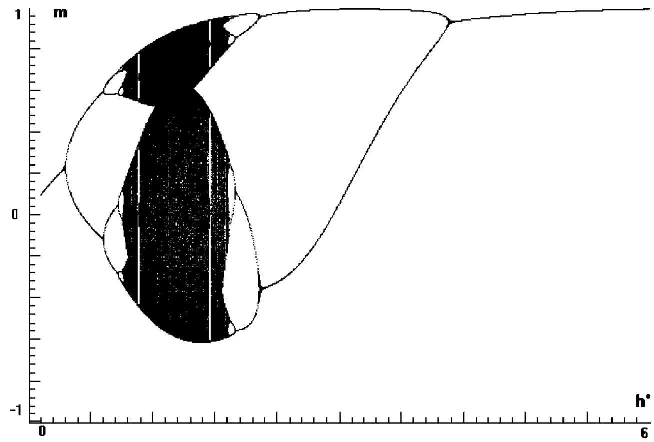

FIG. 1. Plot of m, magnetization, vs h', external magnetic field ($kT=1, J'=-1$, $\gamma = 3$).

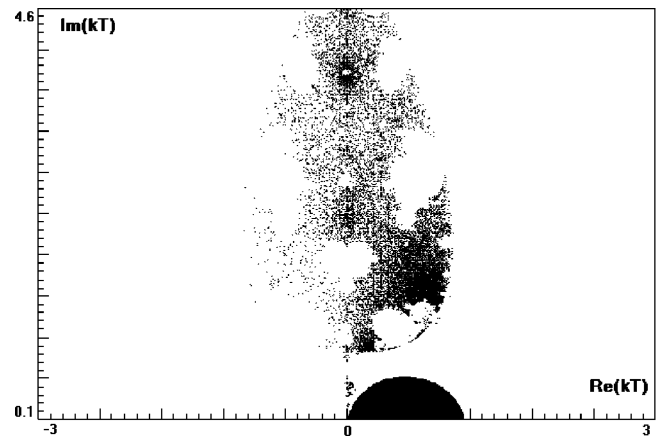

FIG. 2. Complex temperature phase diagram ($h'=3$, $J'=-1$, $\gamma = 3$).

havior, i.e., there takes place a cascade of phase transitions according to the Feigenbaum scheme, Fig. 1. This behavior is a consequence of the fact that attractors of the map (5) have complicated geometrical and dynamical dependence on the values of h and T . In the chaotic region the magnetization is no longer an order parameter and to characterize the three-site interacting Ising model in the chaotic region one should consider the generalized dimensions D_q [5] or Lyapunov exponents λ as the order parameters. For computation of D_q or λ the thermodynamic formalism of the multifractal has been developed [6–8]. In many dynamical systems D_q or λ exhibits nonanalytic behavior that can be interpreted as a phase transition by mapping the problem onto thermodynamics of one-dimensional spin models [9,10]. We recently described the chaotic properties of the three-site interacting Ising model in terms of multifractals and investigated the nonanalytic behavior of λ in the fully developed chaotic region [2,3].

The advantage of Husimi or Bethe-like lattices consists in the fact that they allow one to investigate the behavior of magnetization with the complete set of parameters. The knowledge of the behavior at fixed values of T and h from Eq. (5) would enable the phase structure of the three-site interacting Ising model to be investigated. The information concerning the phase structure, and, in the particular, possible phase transitions for physical values of parameters can be extracted from the behavior of the partition function in the complex plane. The Yang-Lee edge singularity [11] is one example of this, the prediction of the distribution of partition function zeroes in the complex temperature plane is another [12]. It was demonstrated that these beautiful results can be of practical use in the study of phase transitions [13–15].

The phase transition point of the three-site interacting Ising model in the complex plane may be obtained from

$$\mu x_n^{\gamma}+1=0, \tag{7}$$

which expresses the equality of the partition function to zero. Thus, in the thermodynamic limit this condition can be checked on the attractor of the complex map (5). In nonanalytical points this condition gives an unconventional behavior of the magnetization (m). We point out that for real values of parameters the partition function is not equal to zero and the magnetization is not equal to infinity, but if the zeroes of the partition function touch the real axis at some points one can conclude that these are the phase transition points (nonanalytical points of the free energy) .

We verify Eq. (7) on the attractor of the map (5). As an initial condition for the map we take $x_0=1$ (free boundary condition) and for investigation of the attractor's behavior we take $n=10^5 \sim 10^6$ iterations. Thus we require Eq. (7) for at least one x_n of the attractor. The resulting diagrams are shown in Figs. 2 and 3. Our results are stable with respect to the variation of n. In these pictures we draw only the upper part of the complex plane because the partition function of the three-site interacting Ising model has $T \to e^{i\pi}T$ symmetry. One can see that the partition function zeroes lie on a fractal set. The dense region clearly indicates the phase transition condensation. This region disappears at sufficiently high and low external magnetic fields h. The frustration of the three-site interaction on a triangle is the main reason for such condensation. Note that the phase structure of the root site magnetization is slightly different because the Husimi tree is not translation invariant.

A typical example of the condensation of critical points is the Griffiths singularities in the diluted Ising model [16] . The randomness of the interaction constant (or of external

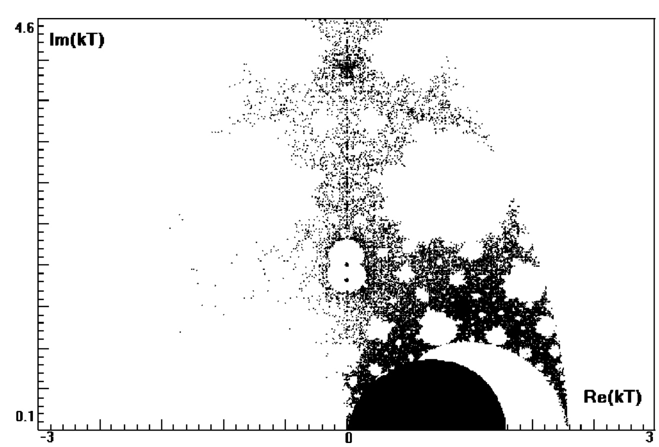

FIG. 3. Complex temperature phase diagram ($h'=3$, $J'=-1$, $\gamma = 4$).

field) gives rise to Griffiths singularities at some specific temperatures, where in the macroscopic region the system is strongly correlated. The frustration in the three-site interact ing Ising model on the Husimi tree causes the appearance of nontrivial thermodynamics and as a consequence in some temperature ranges a different limiting behavior for the magnetization takes place. This fact becomes more important when the condensation of critical points takes place. We believe that the investigation of the recursion relations in terms of multifractal can give us a deeper understanding of the nature of the condensation of critical points.

In summary, we have shown the condensation of critical points in the three-site interacting Ising model on the Husimi tree in the presence of magnetic field. Such behavior of the phase structure is typical for the disordered system obtained here without randomness.

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